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Representations of OSp($M/2n$) and Young supertableaux

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Abstract. Young supertableaux are introduced for finite-dimensional representations of orthosymplectic superalgebras and used to study their content.

1. Introduction

Since the classification of the superalgebras was established [1] their representations have been progressively better understood. In [2] the general theory of the representations was given. The representations of OSp(2|2) and SU(2|1) have been considered in [3] while their general properties were made explicit in [4] for the unitary superalgebras and in [5] for the orthosymplectic ones. Young supertableaux were then proposed [6-9] to describe representations of SU($m|n$). However some ambiguities are present in their definition, due to the existence of representations involving simultaneously covariant and contravariant tensors [9-10]. Such a problem does not appear for orthosymplectic superalgebras since their bosonic part involves orthogonal and symplectic algebras.

Young supertableaux for representations of orthosymplectic superalgebras are defined in this paper, and used to give the content of a representation into representation of its bosonic part SO(M) × Sp($2n$). For such purposes great use is made of the notion of generalised Young tableaux (GYT) introduced in [11] and [12] to perform products of SO(M) representations as well as Sp($2n$) representations.

After recalling in § 2 the main properties of the representations of OSp($M|2n$), we associate in § 3 to each irreducible representation a Young supertableau (YST) and find out the SO(M) × Sp($2n$) representations which compose it. Our method is illustrated in § 4 by two examples and discussed briefly in § 5. Finally some properties of Young tableaux for orthogonal and symplectic groups are recalled in appendix 1.

2. A reminder about the representations of OSp($M|2n$)

There are three kinds of orthosymplectic algebras OSp($M|2n$): $B(m, n) =$ OSp($2m+1|2n$); $C(n) =$ OSp($2|2n-2$), ($n \geq 2$); and $D(m, n) =$ OSp($2m|2n$), ($m \geq 2$,

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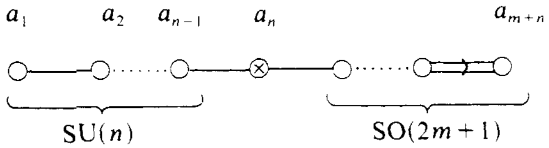
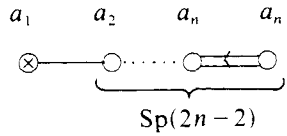
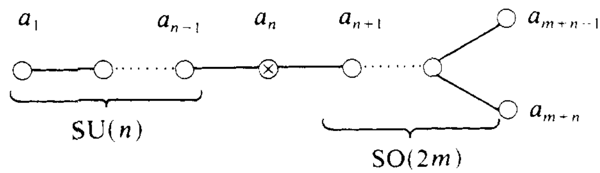
$n > 0$). To each of them one can associate a Kac–Dynkin diagram [1]:



$B(m, n)$

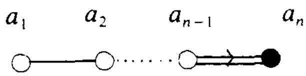


$C(n)$



$D(m, n)$.

We note the special status of $B(0, n) = \mathrm{OSp}(1|2n)$ the Kac–Dynkin diagram of which is



and which carries the property of having only typical representations.

The superalgebra $C(n)$ is of 'type I' while $B(m, n)$ and $D(m, n)$ are of 'type II', the difference being that the odd roots in the first case are in a reducible representation of the bosonic subalgebra $\mathrm{U}(1) \times \mathrm{Sp}(2n-2)$ and in the other cases in an irreducible representation of $\mathrm{SO}(2m) \times \mathrm{Sp}(2n)$ or $\mathrm{SO}(2m+1) \times \mathrm{Sp}(2n)$. The consequence is that a_n in $B(m, n)$ and $D(m, n)$ has to be integer or half-integer, while $a_{n=1}$ can be any complex number in $C(n)$. Let us note that the case of the superalgebras $D(2, 1; \alpha)$ considered as exceptional superalgebras will not be considered in this paper.

An irreducible representation is uniquely characterised by the coordinates of its highest weight Λ in the root space, which appear on the Kac–Dynkin diagram.

The coordinates of Λ in the root space characterise a $\mathrm{SO}(M) \times \mathrm{Sp}(2n)$ representation: the $\mathrm{SO}(M)$ representation can be directly read on the Dynkin diagram, but one of the simple roots (the longest) of $\mathrm{Sp}(2n)$ is 'hidden' behind the odd simple roots. From the knowledge of a_{n+k} $k = 0, 1, \ldots, m$ it is possible to deduce the component b that Λ would have with respect to the longest simple root:

in the $B(m, n)$ case: $b = a_{n\_} - a_{n+1} - \ldots - a_{n+m-1} - \frac{1}{2}(a_{m+n})$ (2.1)

in the $D(m, n)$ case: $b = a_n - a_{n+1} - \ldots - a_{n+m-2} - \frac{1}{2}(a_{m+n-1} + a_{m+n})$. (2.1′)

This immediately implies that the highest weight Λ of a finite $\mathrm{OSp}(M|n)$ representation belongs to an $\mathrm{SO}(M) \times \mathrm{Sp}(2n)$ representation, i.e. one must require $b \geq 0$, and some consistency conditions [2]:

$$\text{for } B(m, n) \text{ if } b < m\text{: } a_{n+b+1} = a_{n+b+2} = \ldots = a_{n+m} = 0 \tag{2.2}$$

$$\text{for } D(m, n) \text{ if } b \leq m-2\text{: } a_{n+b+1} = \ldots = a_{n+m} = 0 \tag{2.2′}$$

and if $b = m - 1$: $a_{m+n-1} = a_{m+n}$.

The OSp($M|2n$) representation is a reducible SO(M)×Sp($2n$) representation, which is obtained from Λ by repeated application of the generators corresponding to the negative odd roots $\beta_j^{a^-}$. (We use here and in the following the notation and results [5].) This deserves a bit of explanation. In OSp($M|2n$), the odd generators belong to the irreducible $(m, 2n)$ representation of SO(M)×Sp($2n$). The distinction between positive and negative odd roots refer to the SO(M)×SU(n) decomposition of the OSp($M|2n$) superalgebra. This decomposition is made such that the OSp($M|2n$) algebra (except $C(n)$) obeys the SU(n)×O(M) gradation:

$$G_{-2}(n(n+1)/2, 1)+G_{-1}(n, M)+G_0[(n^2-1, 1)+(1, M(M-1)/2)] \\ +G_1(\bar{n}, M)+G_2(\overline{n(n+1)}/2, 1).$$

The generators corresponding to the negative odd roots $\beta_j^{a^-}$ belong to G_1 and are in the $(\bar{n}, m)$ representation of SU(n)×SO(M). Denoting $\beta_n^{n^-}$ the SU(n)×SO(M) highest weight in $(\bar{n}, M)$ the other $\beta_j^{a^-}$ are obtained by commuting $\beta_n^{n^-}$ with the generators corresponding to the negative even simple roots. Because of the properties of the algebra, one obtains

$$\{\beta_j^{a^-}, \beta_k^{b^-}\}=\delta_{jk}(T^{ab})^-$$

where $(T^{ab})^-$ are the generators corresponding to the negative roots of Sp($2n$)/SU(n). Applying the odd generators $\beta_j^{a^-}$ on Λ, one builds up new representations of SU(n)× SO(M) and it is only when one applies $(T^{ab})^-\beta_j^{c^-}$ that one recovers SU(n)×SO(M) representations which merge together to build up the Sp($2n$)×SO(M) representation. That means that in the product of $\beta_j^{a^-}$'s one applies on Λ, the symmetric contributions are there to complete the SU(n)×SO(M) representations in the Sp($2n$)×SO(M) representations and it is only the antisymmetric combinations which introduce new Sp($2n$)×SO(M) representations. The resulting OSp($M|2n$) representation is the reducible Sp($2n$)×SO(M) representation obtained by repeated application of $\beta_j^{a^-}$.

It happens sometimes that one might find in an OSp($M|2n$) representation a weight λ, different from the highest weight Λ, but which is annihilated by all the positive even or odd simple roots: the representation is then called atypical, and the Sp($2n$)×SO(M) representations reached from λ are decoupled from the representation. The atypicality of a OSp($M|2n$) representation depends on the coordinates of the highest weight Λ and can be expressed through the atypicality conditions. They are [2]:

for $B(m, n)$:

$$\sum_{t=i}^{n} a_t-\sum_{t=n+1}^{j} a_t+2n-i-j=0 \tag{2.3a}$$

$$\sum_{t=i}^{n} a_t-\sum_{t=n+1}^{j} a_t-2\sum_{t=j+1}^{m+n-1} a_t-a_{m+n}-i+j-2m+1=0 \qquad 1\leqslant i\leqslant n\leqslant j\leqslant m+n-1 \tag{2.3b}$$

for $D(m, n)$:

$$\sum_{t=i}^{n} a_t-\sum_{t=n+1}^{j} a_t+2n-i-j=0 \qquad 1\leqslant i\leqslant n\leqslant j\leqslant m+n-1 \tag{2.4a}$$

$$\sum_{t=i}^{n} a_t-\sum_{t=n+1}^{m+n-2} a_t-a_{m+n}+n-m-i+1=0 \qquad 1\leqslant i\leqslant n \tag{2.4b}$$

$$\sum_{t=i}^{n} a_t - \sum_{t=n+1}^{j} a_t - 2\sum_{t=j+1}^{m+n-2} a_t - a_{m+n-1} - a_{m+n}$$
$$-i+j-2m+2=0 \qquad 1 \leqslant i \leqslant n \leqslant j \leqslant m+n-2 \qquad (2.4c)$$

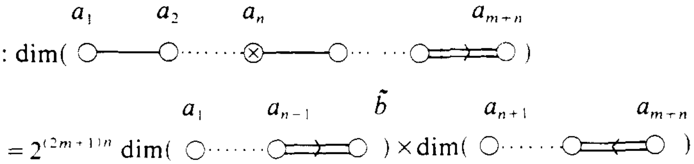
for $C(n)$:

$$a_1 = \sum_{t=2}^{i} a_t + i - 1 \qquad (2.5a)$$

$$a_1 = \sum_{t=2}^{i} a_t + 2\sum_{t=i+1}^{n} a_t + 2n - i - 1. \qquad (2.5b)$$

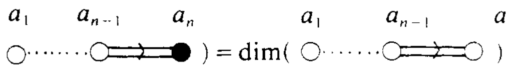
Where none of these conditions are satisfied, the representation is 'typical', and its dimension can be calculated as follows:

for $B(m, n)$:

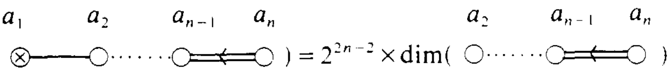


with $\tilde{b} = b - m - \frac{1}{2}$

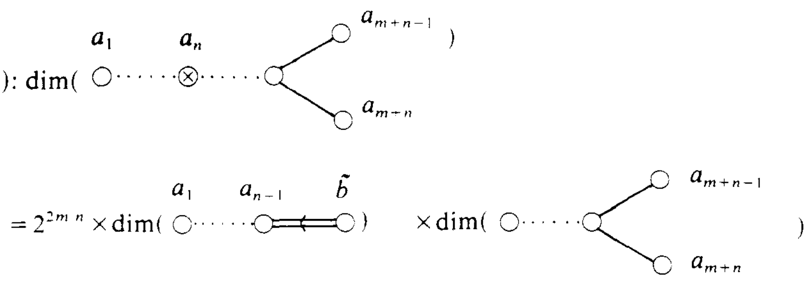
for $B(0, n)$:



for $C(n)$:



for $D(m, n)$:



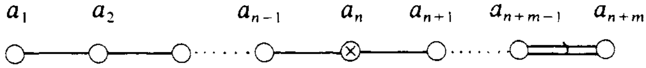
with $\tilde{b} = b - m$.

3. Young supertableaux for OSp($M|2n$)

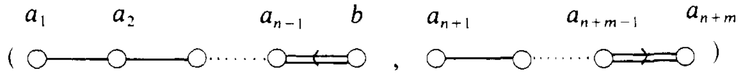
In the same way Young supertableaux were introduced from SU(n) Young tableaux, we construct hereafter Young supertableaux for OSp($M|2n$) algebras with the help of Young tableaux for Sp($2n$) and SO(M) groups. Using techniques developed in [11] and [12] to make products of SO(M) and Sp($2n$) representations (via GYT's), we show in particular how such supertableaux can be used to give the content of a representation of OSp($M|2n$) into representations of Sp($2n$)×SO(M). A brief recall about Young tableaux for Sp($2n$) and SO(M) representations is given in appendix 1.

3.1. *Case of* $B(m, n) = OSp(2m+1|2n)$ *superalgebras*

The following Kac–Dynkin diagram



characterises an IR, the highest weight of which is in the $Sp(2n) \times SO(2m+1)$ IR:



with $b = a_n - a_{n+1} - \ldots - a_{n+m-1} - \frac{1}{2}a_{n+m}$.

We will associate to this $B(m, n)$ representation the Young supertableau (YST): $(\lambda_1, \ldots, \lambda_n; \rho_1, \ldots, \rho_m)$ where:

$$\lambda_i = b + \sum_{t=i}^{n-1} a_t \qquad (i = 1, \ldots, n-1)$$

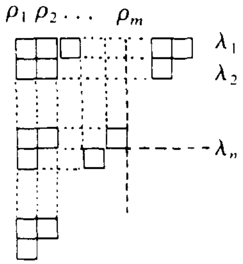
$$\lambda_n = b$$

and (3.1)

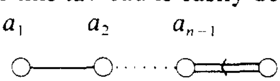
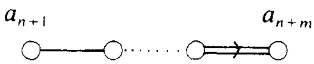
$$\rho_j = n + \tfrac{1}{2}a_{n+m} + \sum_{t=j}^{m-1} a_{n+t}$$

$$\rho_m = n + \tfrac{1}{2}a_{n+m} \qquad (j = 1, \ldots, m-1)$$

and λ_i (resp. ρ_j) has to be seen as the number of of boxes in the ith row (jth column):



Such a tableau is legal if the highest non-vanishing label a_{n+k} is such that $k \leq b$. If $b < m$ one recovers the consistency relations (2.2).

The construction of this tableau is easily done by drawing firstly the $Sp(2n)$ Young tableau associated to  and then adding to the bottom of this tableau the transpose of the Young tableau corresponding to the representation  of $SO(2m+1)$.

Before giving the rule for reducing a $B(m|n)$ representation with respect to its bosonic part, let us introduce the following formal decomposition associated to a YST $\{\mu\}$†

$$\sum_\sigma \left(L_\sigma \times [\mu], \sum_\alpha P_\alpha^2 \times \tilde{L}_\sigma \right) \tag{3.2}$$

† A formula of this type has been proposed, in the form $\langle\mu\rangle \to \Sigma_\sigma \langle\mu/\sigma\rangle, [\tilde{\sigma}/D]$ by King [7]; for the notation see [13].

where the first term in the bracket of the RHS of (3.2) has to be read as relative to Sp($2n$) while the second is relative to SO($2m+1$). The L_σ's are the negative GYT introduced in [11] and the sum is over all the possible ordered partitions σ defining negative YT. In the product $L_\sigma \times [\mu]$ only the positive YTs will be retained. To each L_σ one associates a positive YT $\tilde{L}_\sigma$ obtained by transposing L_σ and considered as an orthogonal Young tableau. The P_α^2 are the negative GYT's introduced in [12] and in the product $P_\alpha^2 \times \tilde{L}_\sigma$ only positive YT's have to be retained (see examples in § 4).

In this decomposition, one has to keep only legal Sp($2n$) and SO($2m+1$) Young tableau. Moreover, each time one gets from the product $P_\alpha^2 \times \tilde{L}_\sigma$ a YT with more than m rows but less than $(2m+1)$ rows, one has to replace it by the 'shortened tableau' obtained by changing each column with p boxes, $2m+1 \geq p > m$, by a column with $2m+1-p$ boxes; of course, if the obtained tableau is no more a YT (i.e. of the form $(\rho_1, \ldots, \rho_m)$ with $\rho_1 \geq \ldots \geq \rho_m \geq 0$) it has to be thrown away. Finally, if two identical YTs appear, the first one obtained by this last rule and the other one right away from the product $P_\alpha^2 \times \tilde{L}_\sigma$, then one must keep only one of them. Note that these concepts on modification rules are conveniently summarised in [13].

Now, let us announce the rule giving the reduction of a $B(m, n)$ representation into Sp($2n$)$\times$SO($2m+1$) ones; we have to separate the typical representation case from the atypical representation case.

3.1.1. Typical representation. Write the highest weight representation $([\lambda], [\kappa])$ of Sp($2n$)$\times$SO($2m+1$) in the representation $\{\mu\} = (\lambda_1, \ldots, \lambda_n; \rho_1, \ldots, \rho_m)$ (see (3.1)) of $B(m, n)$ where the orthogonal Young tableau $[\kappa] = [\kappa_1, \kappa_2, \ldots, \kappa_m]$ with $\kappa_i = \rho_i - n$ denotes the number of boxes in the ith row.

If $[\kappa]$ is the trivial representation, apply the 'decomposition formula' (3.2).

If $[\kappa]$ is not the trivial SO($2m+1$) representation, the decoposition will be:

$$\{\mu\} = \sum_\sigma (L_\sigma \times [\lambda], \tilde{L}_\sigma \times [\kappa]). \tag{3.3}$$

The sum is over all the partitions σ yielding by the product $L_\sigma \times [\lambda]$ legal (and therefore positive) Sp($2n$) Young tableaux. The products $\tilde{L}_\sigma \times [\kappa]$ have to be considered as products of orthogonal Young tableaux (satisfying the rules given in [6]) but we will keep in these products *only* the terms appearing in the decomposition given by expression (3.2). Finally, the tableaux $\tilde{L}_\sigma$ with more than m rows but less than $(2m+1)$ rows will be considered and replaced by their corresponding 'shortened' partner tableau (see above) when possible. We remark that automatically after translating at most $(2m+1)n$ boxes from $[\lambda]$ to $[\kappa]$ the RHS of (3.2) has no meaning in the sense that only illegal SO($2m+1$) YT's would then appear.

3.1.2. Atypical representation. There are $2mn$ atypical conditions for $B(m, n)$ (see (2.3a, b)) but in general some of these conditions are not relevant, requiring values of the Kac–Dynkin labels *a priori* not allowed.

We shall identify an atypical condition by a couple of integers (k, l) $k = 1, \ldots, 2m$; $l = 1, \ldots, n$; where $k \leq m$, $k = j - n + 1$, $l = n - i + 1$, (i, j) appearing in (2.3a) $m > k$, $k = j - n + m + 1$, $l - n - i + 1$ (i, j) in (2.3b). An atypical (k, l) $B(m, n)$ representation can be decomposed into representations of its bosonic part using, firstly, the 'typical' decomposition given above and, next, by taking away at the rth level, r = number of boxes of L_σ, the eventual Sp($2n$)$\times$O($2m+1$) highest weight which may label a

$(k-\beta, l-\alpha)$ atypical $B(m, n)$ representation where $\alpha+\beta=r$. The decoupling has to be performed for the larger $\mathrm{Sp}(2n)\times \mathrm{O}(2m+1)$ weight at the level where the atypical $(k-\beta, l-\alpha)$ representation first appears. We remark that if the decoupling takes place at the first level ($r=1$) one has to take away an atypical representation of the same kind.

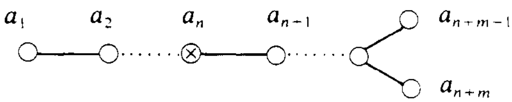
3.1.3. Spinorial representation (for the SO(2m+1) part). The case of $B(m, n)$ representations containing $\mathrm{SO}(2m+1)$ spinorial representation (i.e. a_{n+m} odd number) deserves a bit of attention.

First of all, we remark that such representations are always typical.

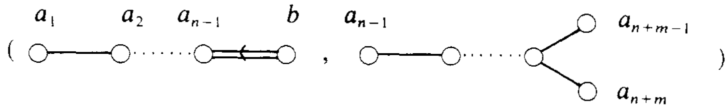
Young tableaux for spinorial representations are defined in appendix 1 and have to be used in (3.3). The supertableau $\{\mu\}$ will now contain □ boxes as well as $\boxed{s}$ boxes (specifically, its $(m+1)$th row being formed by m spinorial boxes). In order to write the RHS of (3.2) in this case it is sufficient to consider $[\mu]$ as made only of □ boxes, that is, to erase the s-label inside the $\boxed{s}$ boxes which are present, and finally to put s-symbols in the m boxes which constitute the first column of $\tilde{L}_\sigma$ before doing the product $P_\alpha^2\times\tilde{L}_\sigma$. Of course this means that one restricts oneself to $\tilde{L}_\sigma$ with at least m rows.

3.2. Case of $D(m, n)=OSp(2m|2n)$ superalgebras

The following Kac–Dynkin diagram:



characterises an IR, the highest weight of which is in the $\mathrm{Sp}(2n)\times\mathrm{SO}(2m)$ IR:



with $b=a_n-a_{n+2}-\ldots-a_{n+m-2}-\frac{1}{2}(a_{n+m-1}+a_{n+m})$.

We will associate to this $D(m, n)$ representation the Young supertableau $(\lambda_1,\ldots,\lambda_n;\rho_1,\ldots,\rho_m)$ where

$$\begin{cases}\lambda_i=b+\sum\limits_{t=i}^{n-1}a_t & (i=1,\ldots,n)\\ \lambda_n=b & (j=1,\ldots,m-1)\end{cases}$$

and (3.4)

$$\begin{cases}\rho_j=n+\frac{1}{2}(a_{n+m}-a_{n+m-1})+\sum\limits_{t=j}^{m-1}a_{n+t}\\ \rho_m=n+\frac{1}{2}(a_{n+m}-a_{n+m-1}).\end{cases}$$

Such a tableau is legal if the highest non-vanishing label a_{n+k} is such that $k\leqslant b$. If $b<m$ one recovers the consistency relations (2.2′).

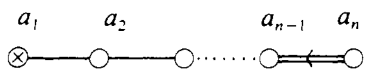
We are in complete analogy with the $B(m, n)$ case and the reduction into bosonic representations can be done using the same techniques to which have to be added the two following prescriptions.

First, if $a_{n+m-1} > a_{n+m}$, leading to a 'non-positive' Young tableau for the SO($2m$) highest weight, one will consider the $D(m, n)$ representation obtained from the previous one by interchanging a_{n+m-1} and a_{n+m}, getting then a positive (conjugate the previous one) Young tableau for the SO($2m$) highest weight. After use of the reduction method, it will only be necessary to replace on the result each SO($2m$) representation by its conjugate to get the decomposition of the $D(m, n)$ representation of interest.

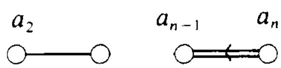
Secondly, one has to stress that the $\tilde{L}_\sigma$ appearing in the RHS of (3.2) and (3.3) has to be seen as an O($2m$) representation. That means that in the RHS of (3.2) when a YTS (or its 'shortened' version) in $P_\alpha^2 \times \tilde{L}_\sigma$ has m rows, it has to be decomposed into the sum of two conjugate SO($2m$) representations (see appendix 1), and in the RHS of (3.3) when $\tilde{L}_\sigma$ (or its 'shortened' version) has m rows, it has also to be decomposed into the sum of two conjugate SO($2m$) representations before performing the product $\tilde{L}_\sigma \times [\kappa]$.

3.3. *Case of $C(n) = OSp(2|2n-2)$ superalgebras*

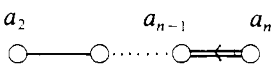
The following Kac–Dynkin diagram:



characterises an IR, the highest weight of which belongs to the U(1)×Sp($2n-2$) representation

(b, ) with $b = a_1 - \sum_{t=2}^{n} a_t$.

The label a_1 can now take any complex value. This arbitrarity on a_1 implies the existence, for any fixed set $(a_2, \ldots, a_n)$ of an infinite class of typical $C(n)$ representations $(a_1, \ldots, a_n)$ with a_1 not satisfying (2.5a, b), of the same dimension $= 2^{2n-2} \times \dim$

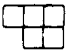
()

In the following we will restrict ourselves to a_1 real and integer.

To the representation $(a_1, \ldots, a_n)$ we will associate the Young supertableau defined as follows: for $b > 0$

$$\lambda_1 = b + (k-2)$$
$$\rho_i = 1 + \sum_{t=i+1}^{n} a_t \qquad (i = 1, \ldots, n-1) \tag{3.5}$$

where k is the index of the last non-vanishing Kac–Dynkin label a_k (i.e. $a_{k+1} = a_{k+2} = \ldots = 0$). As before, λ_1 denotes the number of boxes in the first row and ρ_i the number of boxes in the ith column.

For $b < 0$: we choose to define $\lambda_1 = k - 1$ and ρ_i as given in (3.5) above. However, λ_1 will now denote the number of positive boxes in the first row, which will also contain b negative ones. As an example, the Young supertableau associated with the representation $(a_1=a_2=0,\ a_3=1)$ of $C(3)$ will be:  since in this case $k=3$, $b=-1$ and $\lambda_1=\rho_1=\rho_2=2$. We note that the diagram obtained by erasing the first row is just the transpose of the Young tableau associated to the representation $(a_2, \ldots, a_n)$ of $\mathrm{Sp}(2n-2)$.

The content of a $C(n)$ representation $\{\mu\}$ into its $\mathrm{U}(1)\times\mathrm{Sp}(2n-2)$ parts can now be formed as follows.

For typical representations we have the decomposition formula:

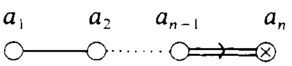
$$\{\mu\}=\sum_{i=0}^{2n-2}\left(b-i,[\tilde{\rho}]\times A_i+\sum_{j=1}^{|i/2|}[\kappa]\times A_{i-2j}\right) \tag{3.6}$$

where in the RHS, the first term refers to U(1) and the second one to $\mathrm{Sp}(2n-2)$. By A_i we denote the completely antisymmetric ith order Young tableau of $\mathrm{Sp}(2n-2)$—that is A_i has only one column of length i, and $|i/2|$ means the entire part of $i/2$. Such products of Young tableaux have to be understood as products of $\mathrm{Sp}(2n-2)$ Young tableaux [12]. However tableaux with l rows, such that $2n-2\geq l>n-1$ appearing in these products have to be retained and replaced by their 'shortened' analogous (see definition given previously) with $(2n-2-l)$ rows, when possible. Finally for $i>n-1$, one has to keep in the sum over j only the terms such that $2n-2-i>i-2j$ or $(n-1)>(i-j)$.

For atypical representations we will again use formula (3.6) but with some modifications. The $(2n-2)$ atypicality conditions are given by (2.5a, b), and we will talk about the jth atypicality condition as the one given by (2.5a) with $i=j$ if $j\leq n-1$ and by (2.5b) with $i=j-(n-1)$ if $j>n-1$. Then considering a $C(n)$ representation satisfying the jth atypicality condition, its decomposition with respect to $\mathrm{U}(1)\times\mathrm{Sp}(2n-2)$ will be given by (3.6) in which will be discarded the obtained $\mathrm{Sp}(2n-2)$ Young tableaux such that the jth row contains one more box than the jth row of $[\kappa]$ if $j>n-1$. In the case of $j>n-1$, then the decomposition will be given by the parts which should be neglected for an atypical representation of the kind $j'=2n-2-j+1\leq n-1$. We remark that the sum of the dimensions of two atypical representations one of kind j and the other of kind $j'=2n-2-j+1$ relative to the same set of labels $(a_2, \ldots, a_n)$ is exactly the dimension of the typical representations labelled by a_k $(k=2, \ldots, n)$.

3.4. *Case of $B(0, n)=OSp(1|2n)$ superalgebras*

As already noticed $B(0, n)$ superalgebras constitute a special case in the sense that all their representations are typical. Moreover these typical representations do not have the same number of even (bosonic) and odd (fermionic) degrees of freedom.

The $\mathrm{Sp}(2n)$ representation containing the highest weight in a $\mathrm{OSp}(1|2n)$ representation can be directly read from the diagram: 

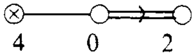
by replacing the odd root a_n by the nth $\mathrm{Sp}(2n)$ root labelled by $b=\frac{1}{2}a_n$. The corresponding Young supertableau $\{\mu\}$ has the shape of a usual Young tableau associated to the $\mathrm{Sp}(2n)$ representation $(a_1, a_2, \ldots, b)$ and its content in terms of $\mathrm{Sp}(2n)$ IRs can be obtained by the formula:

$$\{\mu\}=\sum_{\gamma} L_\gamma\times[\mu]$$

where the sum is over all the partitions $\gamma=0;\ -1;\ -1,\ -1;\ldots;\ -1,\ -1,\ldots,-1$ (n times) and $[\mu]$ is the highest weight Sp$(2n)$ representation.

4. Two illustrative examples of decomposition

We apply hereafter our methd of reduction of an orthosymplectic superalgebra representation with respect to its bosonic part on two specific examples. The first one deals with an atypical representation of $B(2,1)$ while in the second one a 'spinorial' representation of $D(3,1)$ is considered.

A. The representation  of $B(2,1)=\mathrm{OSp}(5|2)$

There are four atypical conditions for $B(2,1)$, namely:

(I) $a_1=0$

(II) $a_1=a_2+1$

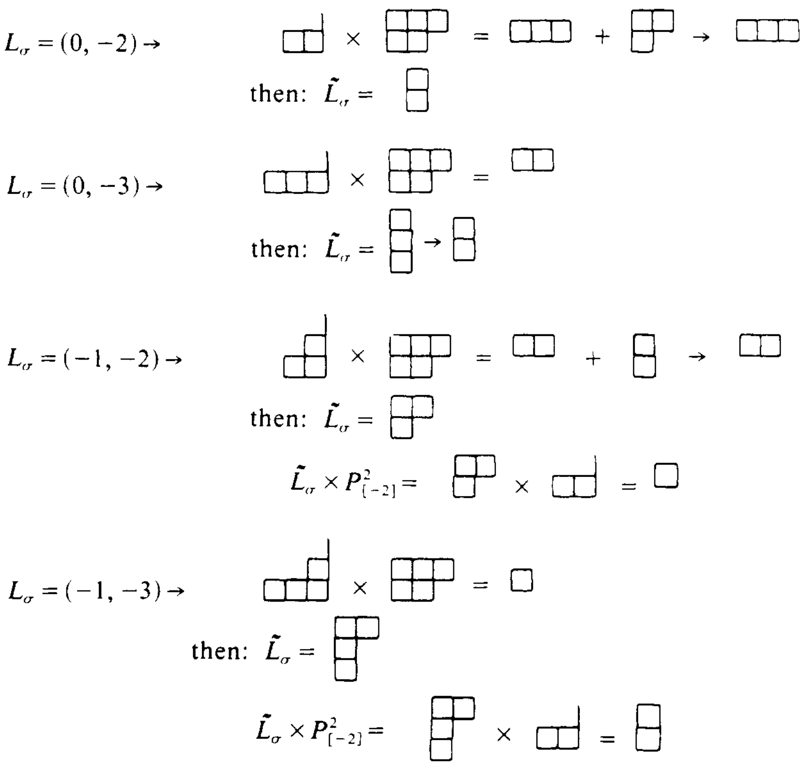
(III) $a_1=2a_2+a_3+2$

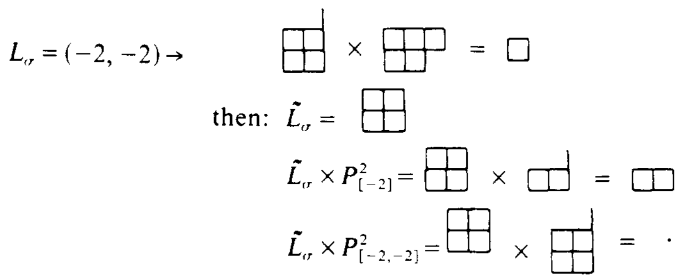
(IV) $a_1=2a_2+a_3+3$.

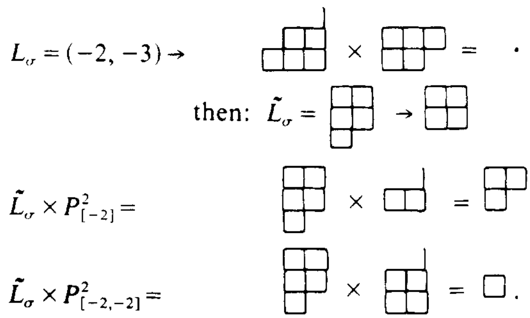
We notice that the representation ($a_1=4$, $a_2=0$, $a_3=2$) satisfies the third condition of atypicality.

From formula (3.1) we have $\lambda_1=b=3$; $\rho_1=\rho_2=2$ and therefore the Young supertableau associated to this representation is: $\{\mu\}=$  while the highest weight representation of Sp(2) × SO(5) is: ().

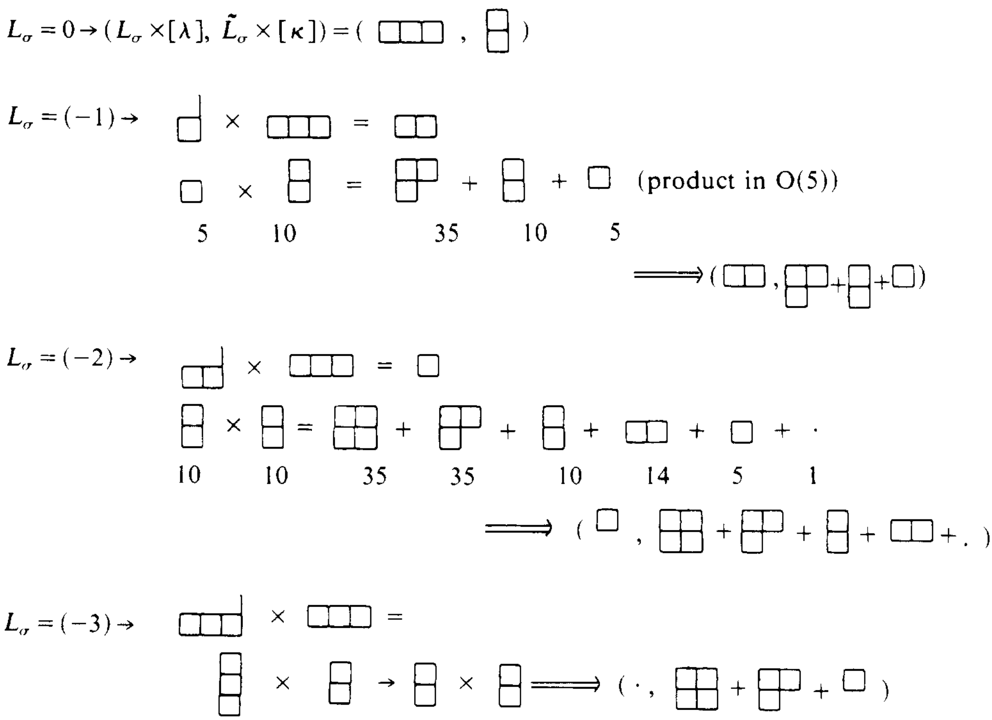
Then considering (3.2) we can start with $L_\sigma=(0,-2)$

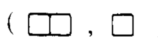
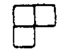




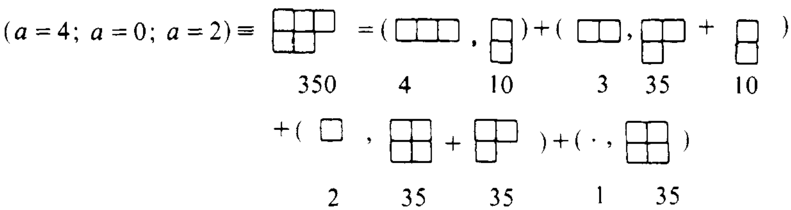


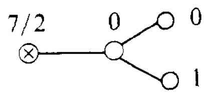
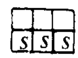
While (3.3) gives, with $[\lambda]=$  and $[\kappa]$ = 



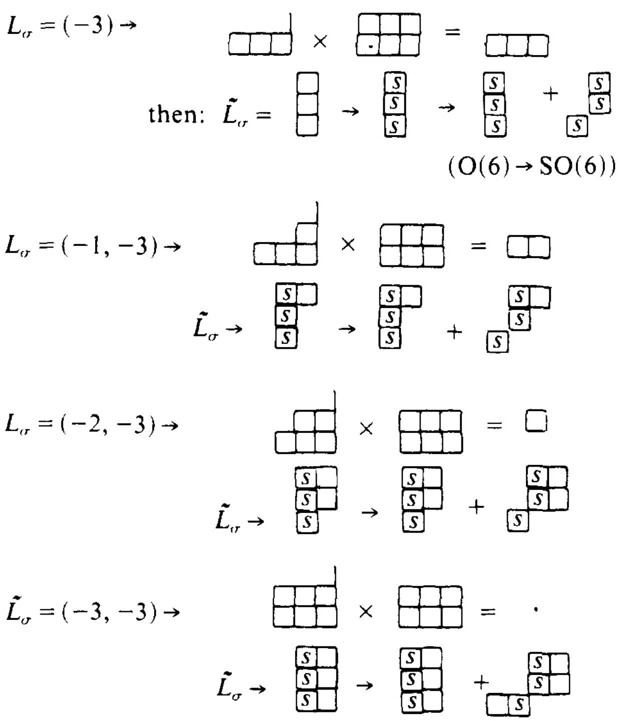
We notice that the Sp($2n$)$\times$SO(5) representation  can be seen as the highest weight of the $B(2, 1)$ representation ($a_1 = 3$, $a_2 = 1$, $a_3 = 0$) which is of the same kind of atypicality (i.e. $a_1 = a_2 + a_3 + 2$) as the representation we want to reduce. Moreover reducing this representation the Young supertableau of which is  with respect to its bosonic part gives (we leave the details to the reader):

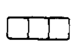
Thus we can notice that this last $B(2, 1)$ representation appears completely in our decomposition: following our rules we have to delete it. Finally we obtain the decomposition of $\{\mu\}$ with respect to $Sp(2) \times SO(5)$:

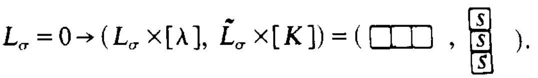


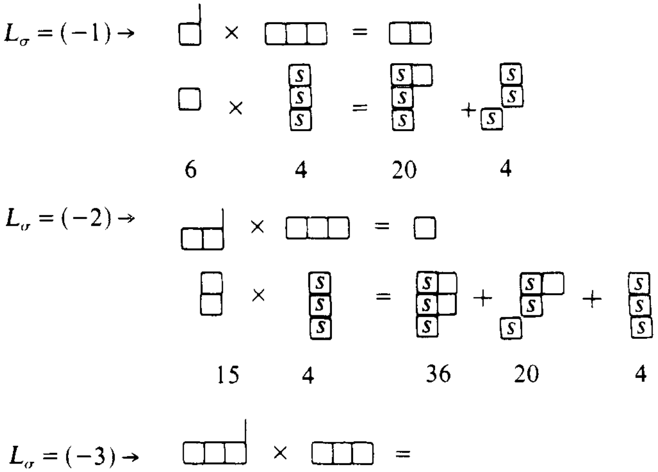
B. The representation  of $D(3, 1) = OSp(6|2)$ then $b = \lambda_1 = 3$ and $\rho_1 = \rho_2 = \rho_3 = \frac{3}{2}$, and the corresponding Young supertableau is $\{\mu\} =$  .

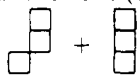
So let us first apply (3.2):

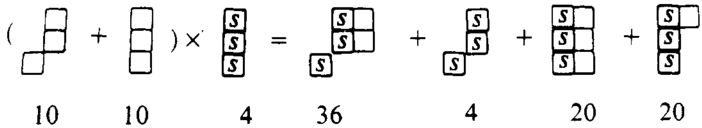


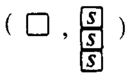
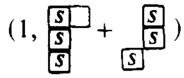
while (3.3) gives, with $[\lambda] =$  and $[K] =$ 

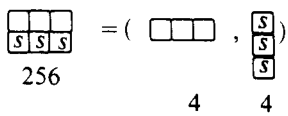


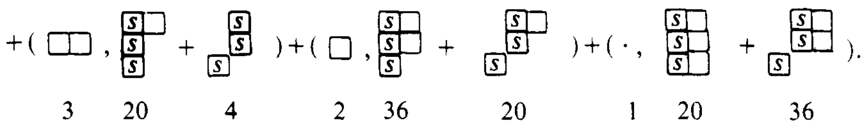


Now $\tilde{L}_\sigma =$  and this O(6) representation decomposes under SO(6) into the two representations , and therefore:



Comparing the pieces given by (3.2) and (3.3) we realise that the Sp(2)×SO(6) representations  as well as  which are present from (3.2) do not appear from (3.2): following our rules, they have to be discarded. Finally we obtain

$(a_1 = \frac{7}{2},\ a_2 = a_3 = 0,\ a_4 = 1) \equiv$ 



5. Role of the odd roots and Young tableaux

Before discussing the method given in § 3, let us make two brief remarks. The first one is relative to the apparent dissymmetry between the way Sp($2n$) and SO(M) tensors are treated: this reflects the property of the generators corresponding to negative odd roots to be in a SU(n)×SO(M) representation, the SU(n) group being included in Sp($2n$). The second point has already be mentioned: it deals with the fact that the consistency conditions of § 2 can be translated as a legality requirement for the Young supertableaux.

Now an important property is the following.

Lemma. $\beta_j^{a^-}$ extracts a box from the Sp($2n$) Young tableau associated with the highest weight Λ.

The proof stands in the fact that the operator measuring the number of boxes k in the Sp($2n$) representation associated with Λ:

$$K = h_1 + 2k_2 + \ldots + nB$$

where

$$B = h_n - (h_{n+1} + \ldots - h_{n+m-1} + \tfrac{1}{2}h_{n+m}) \qquad \text{for } B(m, n)$$

and

$$= h_n - (h_{n+1} + \ldots + \tfrac{1}{2}(h_{n+m-1} + h_{n+m}) \qquad \text{for } D(m, n)$$

and $h_1, \ldots, h_{n+m}$ Cartan generators, acts on any any even root α_a^- and on $\beta_n^{n^-}$ as follows:

$$[K, \alpha_a^-] = 0, \qquad [K, \beta_n^{n^-}] = -\beta_n^{n^-}.$$

Moreover as observed in § 2 only the antisymmetric combination of $\beta_j^{a^-}$ contributes to create new Sp($2n$)×SO(M) highest weights; i.e. the Sp($2n$) indices a have opposite symmetry properties from the SO(M) indices j. The action of $\beta_j^{a^-}$ on the SO(M) indices is more complicated and cannot, in general, be analysed without specifying the SO(M) IR, but what is relevant is the fact that the $\beta_j^{a^-}$ belong to the fundamental M-dimensional representation of SO(M). If the highest weight of SO(M) does correspond to the trivial IR, it gives rise to the 'decomposition' formula. If the highest weight SO(M) IR is not the trivial one, the action of $\beta_j^{a^-}$ can be more easily described, due to the above property, in terms of the product with the dual of the Young tableau which is formed from the boxes taken from Sp($2n$) YT. However, in the product performed according to the rules of SO(M) YTs product, terms not appearing in (3.2) do not appear because they cannot be built up with the $\beta_j^{a^-}$ operators, so this explains the origin of the rule stated in § 3.

An atypicality condition expresses the decoupling of a Sp($2n$)×SO(M) highest weight, i.e. the decoupling for some $\beta_j^{a^-}$ of a $\beta_j^{a^-}\chi$ weight, even if χ belongs to the OSp($M|n$) representation. In this case, in order to obtain IRs described by YSTs we consider that the action of $\beta_j^{a^-}$ on χ gives a vanishing vector. So we are not allowed to apply this fermionic operator. It follows that not all the SO(M) IR, which *a priori* should appear in the $\tilde{L}_\sigma \times [\kappa]$ product of (3.3), are present.

The interpretation of atypical representations in terms of YSTs is as follows.

(a) There are not enough boxes to apply all the necessary antisymmetric combinations of $\beta_j^{a^-}$.

(b) Some 'passages' of boxes from left (Sp($2n$) YT) to right (O(M) YT) are forbidden because their corresponding fermionic lowering operator cannot be used. In this case, using the rules valid for typical representations, we would get invariant subspaces which have to be taken away in order to have the right OSp($M|n$) IR.

Acknowledgments

We are grateful to J P Hurni [14] and H Ruegg for many useful discussions and comments and to J Thierry-Mieg for remarks and also for having provided us with his tables on representations of superalgebras.

Note added in proof. After completion of this work, we found the paper of Farmer and Jarvis [15] which deals with tensor representation of OSp($M|2n$); their methods and ours are closely related.

Appendix. Young tableaux for orthogonal and symplectic groups

An irreducible representation of SO(p) is defined by m non-negative integers $(a_1, \ldots, a_m)$ or Dynkin labels. Separating the cases (i) $p=2m$ and (ii) $p=2m+1$ one can also represent it by p integers or half-integers $(\lambda_1, \ldots, \lambda_p)$ related to the Dynkin labels as follows:

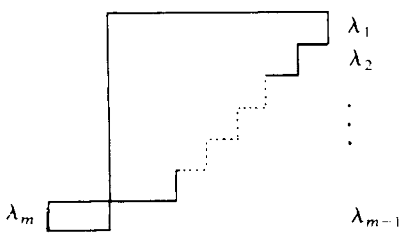
$$\text{(i)}\quad a_j = \lambda_j - \lambda_{j+1} (j = 1, \ldots, p-1) \qquad \text{(ii)}\quad a_j = \lambda_j - \lambda_{j+1}$$
$$a_m = \lambda_{m-1} + \lambda_m \qquad\qquad a_m = \lambda_m. \tag{A1}$$

In case (i) all the λ's are positive or null except λ_m which may be negative, while in case (ii) all the λ's are positive or null. More precisely

$$\text{(i)}\quad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant |\lambda_m| \qquad \text{(ii)}\quad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m \geqslant 0. \tag{A2}$$

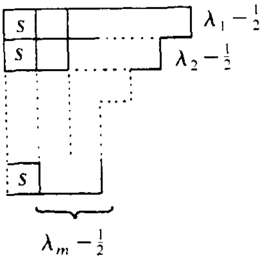
Actually, any irreducible O(p) representation with $\lambda_m = 0$ is irreducible under SO(p). If $\lambda_m \neq 0$, this property is still valid for $p = 2m+1$, while for $p = 2m$ the O($2m$) representation $(\lambda_1, \ldots, \lambda_m)$, $\lambda_m > 0$ splits into the two conjugate SO($2m$) representations $(\lambda_1, \ldots, \lambda_{m-1}, \lambda_m)$ and $(\lambda_1, \ldots, \lambda_{m-1}, -\lambda_m)$. If the λ's are all half-integer, the representation is called spinorial (it is a representation of the covering group of SO(p)).

From the property (A2) one sees that one can associate to a SO(p) representation a Young tableau with λ_i boxes in the ith row if the λ's are all integer and non-negative. In the case where λ_m is negative, a new type of Young tableau—or generalised Young tableau, GYT—has been introduced in [11], [12] the last row of which being called a 'negative row':

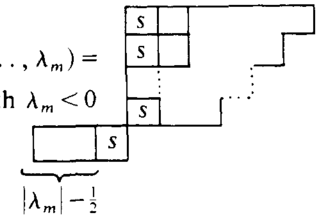


To the spinorial representation $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with λ_i's half-integer was associated in [11] for technical reasons the tableau $(\lambda_1 - \frac{1}{2}, \ldots, \lambda_m - \frac{1}{2})$: we will choose another way to represent it hereafter. Introducing a 'spinorial' box $\boxed{S}$ which can be seen as a 'half' box with respect to the usual one $\square$, the spinorial representation of SO($2m$) or SO($2m+1$) $(\lambda_1, \ldots, \lambda_m)$ with λ_m half-integer will be represented with m $\boxed{S}$ boxes and $(\lambda_1+\ldots+|\lambda_m|-m/2)$ $\square$ boxes as follows:

$(\lambda_1, \ldots, \lambda_m) =$ with $\lambda_m > 0$

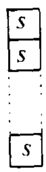


$(\lambda_1, \ldots, \lambda_m) =$ with $\lambda_m < 0$

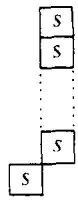


in particular for the fundamental representations:

$(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) =$



$(\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2}) =$



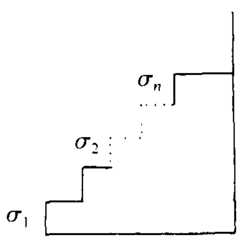
An irreducible representation of Sp($2n$) defined by the Dynkin labels $(a_1, \ldots, a_n)$ can also be represented by a—positive—Young tableau $(\rho_1, \ldots, \rho_n)$ with ρ_i non-negative integers related to the Dynkin labels by

$$a_j = \rho_j - \rho_{j+1}$$
$$a_n = \rho_n \qquad \text{(A3)}$$

and satisfying: $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_n \geqslant 0$.

In order to obtain products of orthogonal and symplectic representations, completely negative Young tableaux have been introduced in [11], [12]. Those appearing in (3.2) and (3.3) are the following

$$L_{\{\sigma\}} = [0, \ldots, 0, -\sigma_k, \ldots, -\sigma_2, -\sigma_1] \equiv$$

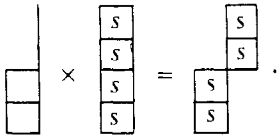


where σ_i's are positive and integer numbers such that $\sigma_i \geqslant \sigma_{i+1}$ and:

$$P^2_{\{\alpha\}} = [0, \ldots, 0, -\alpha_k, \ldots, -\alpha_2, -\alpha_1]$$

where α_i's are positive even numbers such that $\alpha_i \geqslant \alpha_{i+1}$.

Product of a negative Young tableau by a positive one, as well as product of orthogonal and symplectic representations will be carried on using the methods presented in [11], [12]. In order for our notation to be consistent in the product of a negative tableau by a positive one, a negative box acting on a spinorial box will not cancel it, as it would do on a usual one, but will yield a negative $\boxed{s}$ box: as an example in SO(8):



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